Notation and Mappings

Fact (Notation) — Three notations for the same function:

$$y = x^2 + 3 \qquad f(x) = x^2 + 3 \qquad f : x \mapsto x^2 + 3$$

$f(2)$ means the output when the input is 2: $f(2) = 2^2 + 3 = 7$.

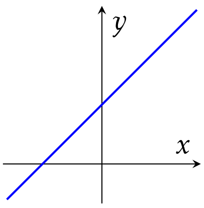
Example

$f(x) = x^2 + 3$. Find $f(-2)$, solve $f(x) = 28$, and simplify $f(2x)$.

$f(-2) = 7 \qquad x^2 + 3 = 28 \implies x = \pm 5 \qquad f(2x) = 4x^2 + 3$

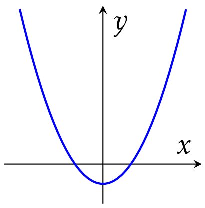
Definition. A **mapping** associates each value of an input set (the **domain**) with values of an output set (the **range**). There are four types:

one-to-one



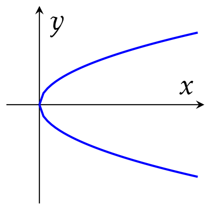
$y = 2x + 3$

many-to-one



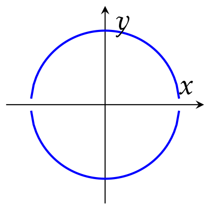
$y = 2x^2 - 1$

one-to-many



$y = \pm\sqrt{x}$

many-to-many



$x^2 + y^2 = 9$

Definition. A **function** is a mapping in which every input has exactly one well-defined output — i.e. a one-to-one or many-to-one mapping.

For example $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is not a function: the $\pm$ allows two outputs.

Example

Classify each mapping ($x \in \mathbb{R}$ unless stated), and state which are functions.

1. $x \mapsto x^3 + 1$
2. $x \mapsto x^2 - 4$
3. $x \mapsto \pm\sqrt{25 - x^2}, \quad -5 \leq x \leq 5$
4. $x \mapsto \cos x, \quad 0° \leq x \leq 360°$

1. one-to-one, function 2. many-to-one, function
3. many-to-many, not a function 4. many-to-one, function

Textbook Exercises: SPS Course 2.10, Exercise 1 Q1–11

Domain and Range

Fact — The domain lives on the x-axis, the range on the y-axis — both are sets of real numbers. State the domain in terms of x and the range in terms of y (or $f(x)$).

To find a range, sketch the graph.

Some inputs must be excluded for the output to exist:

Example

State the largest possible domain of:

1. $f(x) = \sqrt{6x-3}$
2. $g(x) = 2 + \dfrac{3}{2x-5}$
3. $h(x) = \dfrac{\sqrt{x+1}}{x-4}$

1. $x \geq \frac{1}{2}$ *2.* $x \neq \frac{5}{2}$ *3.* $x \geq -1,\ x \neq 4$

Example

Find the range of $f(x) = x^2 + 3x - 1,\ -1 \leq x \leq 1$.

The vertex is at $x = -\frac{3}{2}$, outside the domain, so f is increasing on $[-1, 1]$.
$f(-1) = -3$, $f(1) = 3$: range $-3 \leq y \leq 3$.

Example

Find the range of $f(x) = (x-2)^2 + 3$ for each domain:

1. $x \in \mathbb{R}$
2. $0 \leq x \leq 4$
3. $1 \leq x \leq 5$

Vertex at $(2,3)$.
1. $y \geq 3$
2. Vertex inside; endpoints $f(0) = f(4) = 7$: $3 \leq y \leq 7$
3. Vertex inside; $f(1) = 4$, $f(5) = 12$: $3 \leq y \leq 12$

Example

Find the range of $f(x) = \dfrac{3x-1}{x-2}$, $x \neq 2$.

$$\frac{3x-1}{x-2} = \frac{3(x-2)+5}{x-2} = 3 + \frac{5}{x-2}$$

The fraction $\frac{5}{x-2}$ takes every value except 0, so the range is $y \neq 3$.
(The graph is a hyperbola with asymptotes $x = 2$, $y = 3$.)

Textbook Exercises: SPS Course 2.10, Exercise 1 Q12–13

Composite Functions

Definition. The composite function fg means "do g first, then f":

$$fg(x) = f(g(x)) \qquad x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

The order is right-to-left. $f^2(x)$ means $ff(x)$ — not the same as $(f(x))^2$.

Example

$$f : x \mapsto 2x - 5 \qquad g : x \mapsto x^2 + 2 \qquad h : x \mapsto \frac{1}{x},\ x \neq 0$$

Find $ff(x)$, $fg(x)$, $gf(x)$ and $hgf(x)$. Why does the domain of hgf not exclude any value of x?

$ff(x) = 2(2x - 5) - 5 = 4x - 15$
$fg(x) = 2(x^2 + 2) - 5 = 2x^2 - 1$
$gf(x) = (2x - 5)^2 + 2 = 4x^2 - 20x + 27$
$hgf(x) = \dfrac{1}{4x^2 - 20x + 27}$. *Since $gf(x) = (2x - 5)^2 + 2 \geq 2$, the denominator is never 0.*

Note that $fg \neq gf$ in general.

Example

$f(x) = x^2$ and $g(x) = x + 2$. Solve $fg(x) = gf(x)$.

$(x + 2)^2 = x^2 + 2 \implies 4x + 4 = 2 \implies x = -\frac{1}{2}$

Example

$f(x) = 1 - \dfrac{1}{x}$, $x \neq 0, 1$. Find and simplify $f^2(x)$ and $f^3(x)$, and hence write down $f^{2026}(x)$.

$$f^2(x) = 1 - \frac{1}{1 - \frac{1}{x}} = 1 - \frac{x}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$$

$$f^3(x) = f\left(\frac{1}{1-x}\right) = 1 - (1-x) = x$$

f^3 is the identity, so $f^{2026} = f^{3\times 675+1} = f$: $\quad f^{2026}(x) = 1 - \dfrac{1}{x}$.

Textbook Exercises: SPS Course 2.10, Exercises 2A and 2B

Inverse Functions

Definition. The inverse function f^{-1} returns each output of f to its input: $f^{-1}(f(x)) = x$.
f^{-1} exists only when f is **one-to-one** — otherwise some outputs would return to two inputs.

Tip (Finding an inverse)
Write $y = f(x)$; interchange x and y; make y the subject.

Fact — The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Example

Find the inverse of $f(x) = \dfrac{1}{\sqrt[3]{x+2}}$.

$$x = \frac{1}{\sqrt[3]{y+2}} \implies \sqrt[3]{y+2} = \frac{1}{x} \implies y + 2 = \frac{1}{x^3}$$
$$f^{-1}(x) = \frac{1}{x^3} - 2$$

A many-to-one function can be made one-to-one by restricting its domain.

Example

$f : x \mapsto x^2 - 6x + 16,\ \ x \geq k.$

1. Write f in the form $(x-a)^2 + b$.
2. State the smallest value of k for which f has an inverse.
3. With this k, find f^{-1}, stating its domain and range.

1. $(x-3)^2 + 7$
2. One-to-one to the right of the vertex: $k = 3$.
3. $x = (y-3)^2 + 7 \implies y = 3 + \sqrt{x-7}$ (positive root, since the range of f^{-1} is $y \geq 3$).
$f^{-1}(x) = 3 + \sqrt{x-7}$, domain $x \geq 7$, range $y \geq 3$.
With domain $x \leq 3$ instead, $f^{-1}(x) = 3 - \sqrt{x-7}$.

Example (Edexcel C3)

$$f : x \mapsto 1 - 2x^3,\ x \in \mathbb{R} \qquad g : x \mapsto \frac{3}{x} - 4,\ x > 0$$

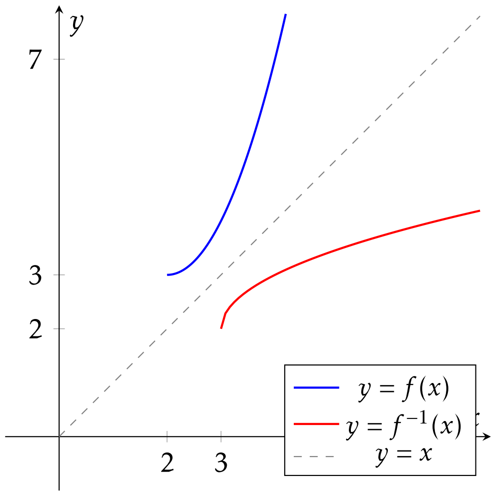
1. Find f^{-1}.
2. Show that $gf : x \mapsto \dfrac{8x^3 - 1}{1 - 2x^3}$.
3. Solve $gf(x) = 0$.

1. $x = 1 - 2y^3 \implies f^{-1}(x) = \sqrt[3]{\dfrac{1 - x}{2}}$
2. $gf(x) = \dfrac{3}{1 - 2x^3} - 4 = \dfrac{3 - 4(1 - 2x^3)}{1 - 2x^3} = \dfrac{8x^3 - 1}{1 - 2x^3}$
3. $8x^3 - 1 = 0 \implies x = \frac{1}{2}$

Textbook Exercises: SPS Course 2.10, Exercise 3A and Exercise 3B Q1–7

Graphs of Inverses

Fact — The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, because reflecting in $y = x$ interchanges x and y.



$f(x) = (x-2)^2 + 3,\ x \geq 2; \qquad f^{-1}(x) = 2 + \sqrt{x-3},\ x \geq 3$

Definition. f is **self-inverse** if $f^{-1} = f$, i.e. $ff(x) = x$ for all x in the domain. Its graph is symmetric about $y = x$.

Example

$$f(x) = \frac{2x-3}{x-2}, \quad x > 2$$

Find the range of f.

2. Show that $ff(x) = x$ for all $x > 2$.

3. Hence write down $f^{-1}(x)$.

1. $f(x) = \dfrac{2(x-2)+1}{x-2} = 2 + \dfrac{1}{x-2}$. *For $x > 2$ the fraction takes every positive value: range $y > 2$.*

2. $ff(x) = \dfrac{2\left(\frac{2x-3}{x-2}\right)-3}{\frac{2x-3}{x-2}-2} = \dfrac{2(2x-3)-3(x-2)}{(2x-3)-2(x-2)} = \dfrac{x}{1} = x$

3. $f^{-1}(x) = \dfrac{2x-3}{x-2}$, $x > 2$: *f is self-inverse.*

Example (Edexcel C3)

$$f : x \mapsto 7x - 1,\ x \in \mathbb{R} \qquad g : x \mapsto \frac{4}{x-2},\ x \neq 2$$

Solve the equation $fg(x) = x$.

2. Hence find the largest value of a such that $g(a) = f^{-1}(a)$.

1. $\frac{28}{x-2} - 1 = x \implies 28 = (x+1)(x-2) \implies x^2 - x - 30 = 0$
   $(x-6)(x+5) = 0 \implies x = 6$ *or* $x = -5$.
2. $g(a) = f^{-1}(a) \iff f(g(a)) = a \iff fg(a) = a$, *so* $a = 6$.

Remark. Solutions of $f(x) = f^{-1}(x)$ usually lie on $y = x$, but not always: a decreasing function can cross its inverse off the line.

Textbook Exercises: SPS Course 2.10, Exercise 3B Q8–20 and Revision Exercise 2.10